Construction of $\mathcal{C}$ operator for a $\mathcal{PT}$ symmetric model

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abstract

We obtain a closed form expression of the $\mathcal{C}(x,y)$ operator for the $\mathcal{PT}$ symmetric Scarf I potential. It is also shown that the eigenfunctions form a complete set.

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In recent years non Hermitian systems, in particular the $\mathcal{PT}$ symmetric ones [1] have been studied widely. Many of these systems are characterised by the fact that they possess real eigenvalues. However for non Hermitian systems the concept of a scalar product is a non trivial one. In fact a straight forward $\mathcal{PT}$ symmetric generalisation of the usual scalar product for Hermitian systems produces a norm which alternates in sign i.e,

$$< \psi_m|\psi_n >_{\mathcal{PT}} = (-1)^n \delta_{mn} \tag{1}$$

With a view to circumvent this difficulty an operator $\mathcal{C}(x,y)$ was introduced [2]. This operator is defined as [2]

$$\mathcal{C}(x,y) = \sum_{n=0}^{\infty} \psi_n(x)\psi_n(y) \tag{2}$$

where $\psi_n(x)$ are eigenfunctions of the Hamiltonian H:

$$H\psi_n(x) = \lambda_n \psi_n(x) \tag{3}$$

However, it is not always easy to obtain a closed form expression of the $\mathcal{C}(x,y)$ operator and often one has to construct it using various approximating techniques [3]. Our purpose here is to obtain a closed form expression of the $\mathcal{C}(x,y)$ operator for a $\mathcal{PT}$ symmetric Scarf I potential.

We consider the Scarf I potential defined by

$$V(x) = \left(\frac{\alpha^2+\beta^2}{2} - \frac{1}{4}\right)\frac{1}{cos^2 x} + \frac{\alpha^2-\beta^2}{2}\frac{sinx}{cos^2 x} \quad , \quad x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \tag{4}$$

where α and β are complex parameters such that $\beta^* = \alpha$ and $\alpha_R > \frac{1}{2}$. In this case the (real) eigenvalues and the corresponding eigenfuctions are given by [5]

$$\begin{aligned} E_n &= \left(n + \tfrac{\alpha+\beta+1}{2}\right)^2 \\ \psi_n(x) &= D_n (1-sinx)^{\frac{\alpha}{2}+\frac{1}{4}}(1+sinx)^{\frac{\alpha^*}{2}+\frac{1}{4}} P_n^{(\alpha,\alpha^*)}(sinx), \qquad n = 0,1,2,...... \end{aligned} \tag{5}$$

where $P_n^{(a,b)}(x)$ denotes the Jacobi polynomial and D_n is a normalisation constant given by

$$D_n = i^n \sqrt{\frac{(2n+2\alpha_R+1)n!\Gamma(n+2\alpha_R+1)}{2^{2\alpha_R+1}\Gamma(n+\alpha+1)\Gamma(n+\alpha^*+1)}} \tag{6}$$

Using the orthogonality properties of Jacobi polynomials [4] it can be shown [5] that the wave functions in (5) satisfy the relation

$$\int_{-\pi/2}^{\pi/2} (\mathcal{PT}\psi_m(x))\psi_n(x)\ dx = (-1)^n \delta_{mn} \tag{7}$$

We now turn to the evaluation of the $\mathcal{C}(x,y)$ operator. Using (5) we obtain from (2)

$$\mathcal{C}(x,y) = \prod_{z=x,y} (1-sinz)^{\frac{\alpha}{2}+\frac{1}{4}}(1+sinz)^{\frac{\alpha^*}{2}+\frac{1}{4}} \sum_{n=0}^{\infty} \frac{(-1)^n(2n+2\alpha_R+1)n!\Gamma(n+2\alpha_R+1)}{2^{2\alpha_R+1}\Gamma(n+\alpha+1)\Gamma(n+\alpha^*+1)} P_n^{(\alpha,\alpha^*)}(sinx)P_n^{(\alpha,\alpha^*)}(siny) \tag{8}$$

To evaluate the summation in (8) we now use the result [6]

$$\sum_{n=0}^{\infty} n! \frac{(2\alpha_R+1)_n}{(\alpha+1)_n(\beta+1)_n}(2n+2\alpha_R+1)P_n^{(\alpha,\alpha^*)}(sinx)P_n^{(\alpha,\alpha^*)}(siny)t^n = \frac{(2\alpha_R+1)(1-t)}{(1+t)^{2\alpha_R+1}}\ F_4(a,b,c,d,U,V) \tag{9}$$

where

$$F_4(a,b,c,d,U,V) = \sum_{r,s=0}^{\infty} \frac{(a)_s(b)_s}{s!(d)_s} \frac{(a+s)_r(b+s)_r}{r!(c)_r} U^r V^s = \sum_{s=0}^{\infty} \frac{(a)_s(b)_s V^s}{s!(d)_s} \ {}_2F_1(a+s,b+s,c,U) \quad (10)$$

$$a = \alpha_R+1, \ \ b = \alpha_R+3/2, \ \ c = 1+\alpha, \ \ d = \beta+1, \ \ U = \frac{(1-sinx)(1-siny)t}{(1+t)^2}, \ \ V = \frac{(1+sinx)(1+siny)t}{(1+t)^2} \quad (11)$$

and ${}_2F_1(a,b,c,z)$ is the standard hypergeometric function.

Now taking the limit $t \to -1$, we obtain

$$F_4(a,b,c,d,U,V) = (-U)^a \sum_{s=0}^{\infty} \frac{\Gamma(c)\Gamma(1/2)}{\Gamma(b+s)\Gamma(c-a-s)} \frac{(-V/U)^s(a)_s(b)_s}{s!(d)_s} \quad (12)$$

Then using (12) we obtain from (9) and (10)

$$\mathcal{C}(x,y) = \mathcal{N}\frac{[(1+sinx)(1+siny)]^{(\alpha^*/2+1/4)}}{[(1-sinx)(1-siny)]^{(\alpha^*/2+3/4)}} \ {}_2F_1(a,1-c+b,d,z), \qquad z = \frac{(1+sinx)(1+siny)}{(1-sinx)(1-siny)} \quad (13)$$

where $\mathcal{N}$ is a constant given by

$$\mathcal{N} = \frac{2\Gamma(\alpha_R+1)sin(\pi(1-c+a))\Gamma(1-c+a)}{\pi \ \Gamma(\alpha^*+1)} \quad (14)$$

It may be noted that (13) is an exact result.

Completeness of the eigenfunctions

The completeness property is very important feature of eigenfunctions. However, to the best of our knowledge for $\mathcal{PT}$ symmetric systems this property has been verified numerically [7]. Here we shall show analytically that the eigenfunctions (5) form a complete set. To do this we note that in a $\mathcal{PT}$ symmetric theory with unbroken $\mathcal{PT}$ symmetry the completeness property can be expressed as [2, 3]

$$\sum_{n=0}^{\infty}(-1)^n\psi_n(x)\psi_n(y) = \delta(x-y) \quad (15)$$

To prove (15) we consider the result [8]

$$\sum_{n=0}^{\infty} \frac{n!\Gamma(a+b+2n+1)\Gamma(a+b+n+1)}{\Gamma(a+n+1)\Gamma(b+n+1)} P_n^{(a,b)}(x)P_n^{(a,b)}(y) = (1+x)^{-b/2}(1-x)^{-a/2}(1+y)^{-b/2}(1-y)^{-a/2}\delta(x-y) \quad (16)$$

where $-1 < x,y < 1, Re(a) > -1, Re(b) > -1$. Now putting $a = \alpha, b = \beta$ in (16) and using (5) we obtain

$$\sum_{n=0}^{\infty}(-1)^n\psi_n(x)\psi_n(y) = \sqrt{cosx \ cosy} \ \delta(sinx - siny) = \delta(x-y) \quad (17)$$

Thus the eigenfunctions (5) form a complete set.

It is interesting to note that two important results can be derived using (16). First we recall that in Hermitian systems, the operator $\mathcal{C}(x,y)$ is just the parity operator i.e., $\mathcal{C}(x,y) = \delta(x+y)$. So for $\alpha = \alpha^*$, (13) should reduce to this limit. Now using the properties of Hypergeometric functions it can be shown that for real $\alpha, \beta, \mathcal{C}(x,y) = \delta(x+y)$. The other properties of the $\mathcal{C}$ operator viz, $\mathcal{C}\psi_n = (-1)^n\psi_n$ follow from the definition (2) and (1) while $\mathcal{C}^2 = 1$ can be derived using the results (7) and (16).

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References

[1] C.M. Bender and S. Boettcher, Phys.Rev.Lett **80**, (1998) 5243;

[2] C.M. Bender, D.C. Brody, and H.F. Jones, Phys.Rev.Lett **89**, (2002) 270401; ibid, **92**, (2004) 119902.

[3] C.M. Bender, P.N. Meisinger and Q. Wang, J.Phys **A36**, (2003) 1973
C.M. Bender and H.F. Jones, Phys.Lett **A328**, (2004) 102
C.M. Bender and B. Tan, J.Phys **A39**, (2006) 1945.

[4] I.S. Gradshteyn and I.M. Ryzhik, Table of Integrals, Series and Products, (Academic Press, 1980).

[5] G. Levai, J.Phys **A39**, (2006) 10161.

[6] W.N. Bailey, Generalised Hypergeometric Series, Cambridge Tract in Math. and Math.Phys.**32** Cambridge University Press, Reprinted by stechert-Hafner Service Agency, New York and London, 1964.
M. Chen and H.M. Srivastava, J.App.Math and Stochastic Analysis, **8**, (1995) 423.

[7] C.M. Bender, S. Boettcher and V.M. Savage, J.Math.Phys **41**, (2000) 6381
C.M. Bender, S. Boettcher, S. Meisinger and Q. Wang, Phys.Lett **A302**, (2002) 286
G.A. Mezincescu, J.Phys **A33**, (2000) 4911.

[8] http://functions.wolfram.com/Polynomials/JacobiP/23/01/